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BLOW-UP PHENOMENA FOR A QUASILINEAR PARABOLIC EQUATION WITH TIME-DEPENDENT COEFFICIENTS UNDER NONLINEAR BOUNDARY FLUX

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Abstract. This paper deals with blow-up phenomena for an initial boundary value problem of a quasilinear parabolic equation with time-dependent coefficient in a bounded star-shaped region under nonlinear boundary flux. Using the auxiliary function method and differential inequality technique, we establish some conditions on time-dependent coefficient and nonlinear functions for which the solution $u(x,t)$ exists globally or blows up at some finite time t^*. Moreover, some upper and lower bounds for t^* are derived in higher dimensional spaces. Some examples are presented to illustrate applications of our results.

1. Introduction

Our main interest lies in the following quasilinear parabolic equation with time dependent coefficient and inner absorption term:

$$u_t = \operatorname{div}(h(|\nabla u|^2)\nabla u) - k(t)f(u), \quad (x,t) \in \Omega \times (0,t^*), \tag{1.1}$$

with the nonlinear Neumann boundary and initial conditions

$$h(|\nabla u|^2)\frac{\partial u}{\partial \nu} = g(u), \quad (x,t) \in \partial\Omega \times (0,t^*), \text{ and} \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \Omega, \tag{1.3}$$

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where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded star-shaped region with smooth boundary $\partial\Omega$ and ν is the unit outward normal vector on $\partial\Omega$. The coefficient $k(t)$ is a nonnegative differentiable function, t^* is a possible blow-up time when blow-up occurs, otherwise $t^* = +\infty$, and the diffusion coefficient $h(\sigma)$ is a C^1-function satisfying the condition

$$h(\sigma) + 2\sigma h'(\sigma) > 0, \quad \sigma > 0, \tag{1.4}$$

so that $\operatorname{div}(h\nabla u)$ is elliptic. The nonlinear functions $f(u)$ and $g(u)$ are nonnegative continuous functions which satisfy appropriate conditions, and the initial data $u_0(x)$ is a positive C^1-function and satisfies a compatibility condition. By the classical parabolic theory, one can deduce that the solution of (1.1)-(1.3) is nonnegative and smooth.

Many physical phenomena and biological species theories, such as the concentration of diffusion of some non-Newton fluid through porous medium, the density of some biological species, and heat conduction phenomena, have been formulated as parabolic equation (1.1), see [3, 16, 25]. The nonlinear boundary flux (1.2) can be physically interpreted as the nonlinear radial law, see [9, 13].

In the past decades, there have been many works dealing with existence and nonexistence of global solutions, blow-up of solutions, bounds for blow-up time, blow-up rates, blow-up sets, and asymptotic behavior of the solutions to nonlinear parabolic equations, refer to [22, 23, 24] and the survey papers [2, 10, 12]. Roughly, it has been seen that existence of global and nonglobal solutions and behavior of the solutions to semilinear parabolic equations depend on nonlinearity, dimension, initial data, and nonlinear boundary flux. In this paper, we would like to investigate whether the solutions of quasilinear parabolic equations blow up and when blow-up occurs. A variety of methods have been used to study the problem above (cf. [11]), and in many cases, the methods used to show blow-up of solutions often provide an upper bound for the blow-up time. However, lower bounds for blow-up time may be harder to be determined. One can refer to [5, 6, 14, 20, 21] to see some studies on the initial-boundary value problem of a parabolic equation without time-dependent coefficients. Payne *et al.* [21] considered the following quasilinear parabolic equation:

$$u_t = \operatorname{div}\left(h(|\nabla u|^2)\nabla u\right) + f(u), \quad (x,t) \in \Omega \times (0, t^*), \tag{1.5}$$

with homogeneous Dirichlet boundary condition, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. They obtained lower bounds for blow-up time by using the technique of differential inequality. The lower bound of the solution to (1.5) with Robin boundary condition was obtained by Li *et al.* [14]. However, under this boundary condition, the best constant of the Sobolev inequality used in [21] is no longer applicable. They impose suitable conditions on h and determined a lower bound for the blow-up time when blow-up occurs in $\mathbb{R}^3$. Enche [5] discussed the quasilinear parabolic equation

$$u_t = \operatorname{div}(h(u)\nabla u) + f(u), \quad (x,t) \in \Omega \times (0, t^*),$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. By virtue of a first-order differential inequality technique, the author introduced some sufficient conditions for which the solution exists globally or blows up. In addition, a lower bound for the blow-up time was also obtained when blow-up occurs. Payne *et al.* [20] studied the quasilinear parabolic equation

$$u_t = \operatorname{div}\left(|\nabla u|^{2p} \nabla u\right), \quad (x,t) \in \Omega \times (0, t^*),$$

with the Neumann boundary condition

$$|\nabla u|^{2p} \frac{\partial u}{\partial \nu} = f(u), \quad (x,t) \in \partial\Omega \times (0, t^*),$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. They derived some lower and upper bounds of blow-up time under some appropriate restrictive conditions. Recently, Fang and Chai [6] considered the quasilinear parabolic equation with an inner absorption term

$$u_t = \operatorname{div}\left((|\nabla u|^p + 1)\nabla u\right) - f(u), \quad (x,t) \in \Omega \times (0, t^*),$$

under the nonlinear boundary flux, where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a sufficiently smooth bounded domain. Using a first-order differential inequality technique, they introduced some sufficient conditions for which the solution exists globally or blows up. Moreover, a lower bound for the blow-up time, when blow-up occurs, was also obtained in $\mathbb{R}^3$. In addition, to see the lower bounds of blow-up time for a semilinear equation with inner absorption term in a higher dimensional space under nonlinear boundary flux or a porous medium equation with nonlocal source and inner absorption, refer to [1, 7].

For the studies on lower bound of blow-up time for a parabolic equation with time-dependent coefficients, one can refer to [8, 15, 17, 18, 19]. Payne and Philippin [18, 19] considered initial-boundary value problems of the semilinear parabolic equation with time-dependent coefficient

$$u_t = \Delta u + k(t) f(u), \quad (x,t) \in \Omega \times (0, t^*),$$

under homogeneous Dirichlet and Neumann boundary conditions, where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a sufficiently smooth bounded domain. Under some conditions on $k(t)$ and $f(u)$, they obtained some sufficient conditions for existence of the global solution and upper bound for the blow-up time. Moreover, a lower bound of the blow-up time was derived in $\mathbb{R}^3$. One can refer to [17] for details on the problem with Robin boundary condition. Recently, Fang and Wang [8] investigated the divergence form of a parabolic equation with time-dependent coefficient and inner absorption term

$$u_t = \sum_{i,j=1}^{N} \left(a^{ij}(x)u_{x_i}\right)_{x_j} - k(t)f(u), \quad (x,t) \in \Omega \times (0,t^*),$$

under nonlinear boundary flux, where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded star-shaped region with smooth boundary $\partial\Omega$. They established some conditions on initial data which guarantee blow-up or global existence of the solutions and derived an upper bound of the blow-up time. They also obtained a lower bound of blow-up time under more restrictive conditions. Marras and Piro [15] studied the quasilinear parabolic equation with time-dependent coefficient

$$u_t = \operatorname{div}(h(|\nabla u|^2)\nabla u) + k(t)f(u), \quad (x,t) \in \Omega \times (0,t^*),$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$. Under some conditions on the initial data and geometry of the spatial domain, they obtained upper and lower bounds of the blow-up time. Moreover, the sufficient conditions for existence of global solution were derived.

In view of the works mentioned above, one can find that research on the blow-up phenomena of the solutions to parabolic equations with absorption terms having time-dependent coefficients under nonlinear boundary flux has not been started yet. A difficulty lies in finding an influence of $k(t)$ and a competitive relationship between inner absorption and boundary source in determining blow-up of the solutions. By virtue of the modified differential inequality technique, we establish some conditions on time-dependent coefficient and nonlinear functions for which the solution $u(x,t)$ exists globally or blows up at some finite time t^*, and we also derive lower and upper bounds for t^* in higher dimensional spaces.

The rest of our paper is organized as follows: In Sections 2 and 3, we establish some conditions on $k(t)$, $h(\sigma)$, $f(u)$, and $g(u)$ for which the solution $u(x,t)$ exists globally or blows up in finite time t^*, and then obtain an upper bound for t^*. A lower bound of t^* is derived in Section 4. Finally, some examples are presented to illustrate applications of our results.

2. The global existence

In this section, we establish some conditions on $k(t)$ and the nonlinear functions f, g, and h to guarantee existence of global solution. In order to prove our results, we first recall a lemma in [20] and state it below:

LEMMA 2.1. [20] *Suppose that Ω is a bounded star-shaped region in $\mathbb{R}^N$, where $N \geq 2$. Then for any nonnegative C^1-function u and constant $\theta > 0$, we have the inequality*

$$\int_{\partial\Omega} u^\theta dS \leq \frac{N}{\rho_0}\int_\Omega u^\theta dx + \frac{\theta d}{\rho_0}\int_\Omega u^{\theta-1}|\nabla u| dx,$$

where

$$\rho_0 = \min_{x\in\partial\Omega}(x\cdot\nu), \quad d = \max_{x\in\overline{\Omega}}|x|.$$

THEOREM 2.2. *Suppose that $h(\sigma)$ is a C^1-function which satisfies the inequality*

$$h(\sigma) \geq b_1\sigma^p + b_2, \quad \sigma > 0, \tag{2.1}$$

with $b_1, b_2, p > 0$ and that the nonnegative functions f and g satisfy the inequalities

$$f(s) \geq a_1 s^q, \quad s \geq 0, \tag{2.2}$$

$$g(s) \leq a_2 s^r, \quad s \geq 0, \tag{2.3}$$

where $a_1 > 0$, $a_2 \geq 0$, $q, r > 1$, $q + 1 > 2r$, and

$$k(t) > 0, \quad \frac{k'(t)}{k(t)} \leq -\beta, \quad t > 0, \tag{2.4}$$

for a positive constant β.

Then the nonnegative solution $u(x,t)$ of problem (1.1)-(1.3) does not blow up; that is, $u(x,t)$ exists for all $t > 0$.

REMARK 2.3. From the conditions $q, r > 1$ and $q+1 > 2r$ in Theorem 2.2, one can easily see that $q > r$.

Proof of Theorem 2.2. Set

$$\Phi(t) = k(t)\int_\Omega u^2 dx. \tag{2.5}$$

We first show that Φ is not an increasing function. To this end, we compute the derivative

$$\Phi'(t) = k'(t)\int_\Omega u^2 dx + 2k(t)\int_\Omega u u_t dx$$

$$= \frac{k'(t)}{k(t)}\Phi(t) + 2k(t)\int_\Omega u\left[\operatorname{div}(h(|\nabla u|^2)\nabla u) - k(t)f(u)\right]dx.$$

By applying (2.1)-(2.4) and the divergence theorem, we have the inequality

$$\begin{aligned}\Phi'(t) \le &-\beta\Phi(t) - 2b_1k(t)\int_\Omega |\nabla u|^{2(p+1)}dx - 2b_2k(t)\int_\Omega |\nabla u|^2dx \\ &+ 2a_2k(t)\int_{\partial\Omega} u^{r+1}dS - 2a_1k^2(t)\int_\Omega u^{q+1}dx.\end{aligned}\tag{2.6}$$

By Lemma 2.1, one can have the inequality

$$\int_{\partial\Omega} u^{r+1}dS \le \frac{N}{\rho_0}\int_\Omega u^{r+1}dx + \frac{(r+1)d}{\rho_0}\int_\Omega u^r|\nabla u|dx.\tag{2.7}$$

It follows from Hölder's and Young's inequalities that

$$\begin{aligned}\int_\Omega u^r|\nabla u|dx &\le \left(\int_\Omega u^{2r}dx\int_\Omega |\nabla u|^2dx\right)^{\frac{1}{2}} \\ &\le \frac{1}{2\varepsilon_1}\int_\Omega u^{2r}dx + \frac{\varepsilon_1}{2}\int_\Omega |\nabla u|^2dx,\end{aligned}\tag{2.8}$$

where ε_1 is a positive constant to be determined later. Substituting (2.7) and (2.8) into (2.6), we get the inequality

$$\begin{aligned}\Phi'(t) \le &-\beta\Phi + \left[\frac{a_2(r+1)d\varepsilon_1}{\rho_0} - 2b_2\right]k(t)\int_\Omega |\nabla u|^2dx + \frac{2a_2N}{\rho_0}k(t)\int_\Omega u^{r+1}dx \\ &+ \frac{a_2(r+1)d}{\rho_0\varepsilon_1}k(t)\int_\Omega u^{2r}dx - 2a_1k^2(t)\int_\Omega u^{q+1}dx.\end{aligned}\tag{2.9}$$

Selecting $\varepsilon_1 = \dfrac{2b_2\rho_0}{a_2(r+1)d} > 0$, the second term in (2.9) vanishes and we obtain the inequality

$$\begin{aligned}\Phi'(t) \le &-\beta\Phi + \frac{2a_2N}{\rho_0}k(t)\int_\Omega u^{r+1}dx + \frac{a_2(r+1)d}{\rho_0\varepsilon_1}k(t)\int_\Omega u^{2r}dx \\ &- 2a_1k^2(t)\int_\Omega u^{q+1}dx.\end{aligned}\tag{2.10}$$

We now focus our attention on $k(t)\int_\Omega u^{2r}dx$. From Hölder's inequality, we can have the inequality

$$k(t)\int_\Omega u^{2r}dx \le \left[k(t)\int_\Omega u^{r+1}dx\right]^\mu \left[k(t)\int_\Omega u^{q+1}dx\right]^{1-\mu},\tag{2.11}$$

where

$$\mu = \frac{q+1-2r}{q-r} \in (0,1).$$

Furthermore, one can obtain the inequalities

(2.12)

$$\begin{aligned} k(t)\int_\Omega u^{2r}dx &\le \left[\varepsilon_2^{\frac{\mu-1}{\mu}} k(t)\int_\Omega u^{r+1}dx\right]^{\mu}\left[\varepsilon_2 k(t)\int_\Omega u^{q+1}dx\right]^{1-\mu} \\ &\le \mu\varepsilon_2^{\frac{\mu-1}{\mu}} k(t)\int_\Omega u^{r+1}dx + (1-\mu)\varepsilon_2 k(t)\int_\Omega u^{q+1}dx, \end{aligned}$$

for arbitrary $\varepsilon_2 > 0$ by the arithmetic and geometric inequality

$$a^s b^{1-s} \le as + b(1-s) \text{ for } a,\ b>0,\ 0<s<1. \tag{2.13}$$

Substituting (2.12) into (2.10) yields the inequalities

(2.14)

$$\begin{aligned} \Phi'(t) \le& -\beta\Phi + \frac{2a_2N}{\rho_0}k(t)\int_\Omega u^{r+1}dx - 2a_1k^2(t)\int_\Omega u^{q+1}dx \\ &+ \frac{a_2(r+1)d}{\rho_0\varepsilon_1}\left[\mu\varepsilon_2^{\frac{\mu-1}{\mu}} k(t)\int_\Omega u^{r+1}dx + (1-\mu)\varepsilon_2 k(t)\int_\Omega u^{q+1}dx\right] \\ \le& M_1 k(t)\int_\Omega u^{r+1}dx - M_2 k(t)\int_\Omega u^{q+1}dx, \end{aligned}$$

where

$$\begin{aligned} M_1 &= \frac{2a_2N}{\rho_0} + \frac{a_2(r+1)d}{\rho_0\varepsilon_1}\mu\varepsilon_2^{\frac{\mu-1}{\mu}} > 0, \\ M_2 &= 2a_1k(t) - \frac{a_2(r+1)d}{\rho_0\varepsilon_1}(1-\mu)\varepsilon_2, \end{aligned}$$

and $\epsilon_2 > 0$ is a sufficiently small constant such that $M_2 > 0$.

An application of Hölder's inequality leads to

$$k(t)\int_\Omega u^{r+1}dx \le k(t)\left(\int_\Omega u^{q+1}dx\right)^{\frac{r+1}{q+1}}|\Omega|^{\frac{q-r}{q+1}}, \tag{2.15}$$

where $|\Omega| = \displaystyle\int_\Omega dx$ is the N-volume of Ω. Inserting (2.15) into (2.14), we have

(2.16)

$$\begin{aligned} \Phi'(t) &\le M_1 k(t)|\Omega|^{\frac{q-r}{q+1}}\left(\int_\Omega u^{q+1}dx\right)^{\frac{r+1}{q+1}} - M_2 k(t)\int_\Omega u^{q+1}dx \\ &= M_1|\Omega|^{\frac{q-r}{q+1}}\left[k(t)\left(\int_\Omega u^{q+1}dx\right)^{\frac{r+1}{q+1}}\right]\left[1 - |\Omega|^{\frac{r-q}{q+1}}\frac{M_2}{M_1}\left(\int_\Omega u^{q+1}dx\right)^{\frac{q-r}{q+1}}\right]. \end{aligned}$$

By using Hölder's inequality, we can have the inequality

$$\Phi(t) = k(t)\int_\Omega u^2 dx \le k(t)\left(\int_\Omega u^{q+1}dx\right)^{\frac{2}{q+1}} |\Omega|^{\frac{q-1}{q+1}},$$

i.e.,

$$\int_\Omega u^{q+1}dx \ge \left[|\Omega|^{\frac{1-q}{q+1}} k^{-1}(t)\Phi(t)\right]^{\frac{q+1}{2}}. \tag{2.17}$$

It follows from (2.16) and (2.17) that

$$\Phi'(t) \le M_1|\Omega|^{\frac{q-r}{q+1}}\left[k(t)\left(\int_\Omega u^{q+1}dx\right)^{\frac{r+1}{q+1}}\right]\left[1-|\Omega|^{\frac{r-q}{2}}\frac{M_2}{M_1}k^{\frac{r-q}{2}}(t)\Phi^{\frac{q-r}{2}}\right], \tag{2.18}$$

with $\frac{q-r}{2} > 0$. Since $\frac{k'(t)}{k(t)} \le -\beta$ and the positive coefficient $k(t)$ is a non-increasing function, one can conclude from (2.18) that $\Phi(t)$ is bounded for all $t > 0$ under the conditions in Theorem 2.2. In fact, if $u(x,t)$ blows up at finite time t^*, then $\Phi(t)$ is unbounded near t^*, which forces $\Phi'(t) \le 0$ in some interval $[t_0, t^*)$, and hence, we have $\Phi(t) \le \Phi(t_0)$ in $[t_0, t^*)$, which implies that $\Phi(t)$ is bounded in $[t_0, t^*)$. This is a contradiction. Therefore, $u(x,t)$ exists for all $t > 0$, which completes the proof. □

For the special case $r = 1$, one can obtain the same result under slightly different conditions.

THEOREM 2.4. *Suppose that the nonlinear functions h, f, and g satisfy (2.1), (2.2), and (2.3) with constants*

$$b_1, b_2, a_1, p > 0, \quad a_2 \ge 0, \quad q > 1, \quad r = 1, \tag{2.19}$$

and

$$k(t) > 0, \quad \frac{k'(t)}{k(t)} \le \beta, \quad t > 0, \tag{2.20}$$

where β is a nonnegative constant.

Then the nonnegative solution $u(x,t)$ of problem (1.1)-(1.3) does not blow up; that is, $u(x,t)$ exists for all $t > 0$.

Proof. Set

$$\phi(t) = k^{\frac{2}{q-1}}(t)\int_\Omega u^2 dx.$$

By using similar arguments as in the proof of Theorem 2.2, we can have the inequality

$$\phi'(t) \leq \frac{2\beta}{q-1}\phi(t) - 2b_2 k^{\frac{2}{q-1}}(t)\int_\Omega |\nabla u|^2 dx + \frac{2Na_2}{\rho_0}k^{\frac{2}{q-1}}(t)\int_\Omega u^2 dx + \frac{4a_2 d}{\rho_0}k^{\frac{2}{p-1}}(t)\int_\Omega u|\nabla u| dx - 2a_1 k^{\frac{q+1}{q-1}}(t)\int_\Omega u^{q+1}dx. \tag{2.21}$$

It follows from Hölder's inequality that

$$\begin{aligned} k^{\frac{q+1}{q-1}}(t)\int_\Omega u^{q+1}dx &\geq |\Omega|^{\frac{1-q}{2}}\left[k^{\frac{2}{q-1}}(t)\int_\Omega u^2 dx\right]^{\frac{q+1}{2}} \\ &= |\Omega|^{\frac{1-q}{2}}\phi^{\frac{q+1}{2}}(t). \end{aligned} \tag{2.22}$$

Let us consider the fourth term on the right side of (2.21). By Hölder's inequality and (2.13), we can obtain the inequalities

$$\begin{aligned} &\frac{4a_2 d}{\rho_0}k^{\frac{2}{q-1}}(t)\int_\Omega u|\nabla u|dx \\ &\leq \frac{4a_2 d}{\rho_0}\left[k^{\frac{2}{q-1}}(t)\int_\Omega u^2 dx\right]^{\frac{1}{2}}\left[k^{\frac{2}{q-1}}(t)\int_\Omega |\nabla u|^2 dx\right]^{\frac{1}{2}} \\ &\leq \frac{4a_2 d}{\rho_0}\left[\frac{1}{2\varepsilon_3}k^{\frac{2}{q-1}}(t)\int_\Omega u^2 dx + \frac{\varepsilon_3}{2}k^{\frac{2}{q-1}}(t)\int_\Omega |\nabla u|^2 dx\right] \\ &\leq \frac{4a_2 d}{\rho_0}\left[\frac{1}{2\varepsilon_3}\phi(t) + \frac{\varepsilon_3}{2}k^{\frac{2}{q-1}}(t)\int_\Omega |\nabla u|^2 dx\right], \end{aligned} \tag{2.23}$$

where ε_3 is a positive constant to be determined later. Combining (2.22) and (2.23) with (2.21), we have the inequality

$$\phi'(t) \geq \left(\frac{2\beta}{q-1} + \frac{2Na_2}{\rho_0} + \frac{2a_2 d}{\varepsilon_3\rho_0}\right)\phi(t) + \left(\frac{2a_2 d\varepsilon_3}{\rho_0} - 2b_2\right)k^{\frac{2}{q-1}}(t)\int_\Omega |\nabla u|^2 dx - 2a_1|\Omega|^{\frac{1-q}{2}}\phi^{\frac{q+1}{2}}(t), \tag{2.24}$$

and hence, taking $\varepsilon_3 = \frac{b_2\rho_0}{a_2 d}$, we get the inequality

$$\phi'(t) \leq c_1\phi(t) - c_2\phi^{\frac{q+1}{2}}(t) = c_1\phi(t)\left(1 - \frac{c_2}{c_1}\phi^{\frac{q-1}{2}}(t)\right), \tag{2.25}$$

where

$$c_1 = \frac{2\beta}{q-1} + \frac{2Na_2}{\rho_0} + \frac{2a_2 d}{\varepsilon_3\rho_0} > 0, \quad \text{and}$$

$$c_2 = 2a_1|\Omega|^{\frac{1-q}{2}} > 0.$$

By an analogous analysis as in the proof of Theorem 2.2, one can easily conclude that the solution $u(x,t)$ exists for all $t > 0$, which completes the proof. □

3. Blow-up and upper bound of t^*

In this section, Ω needs not to be star-sharped. We assume some conditions to assure that the solution of (1.1)-(1.3) blows up at finite time t^* and derive an upper bound for t^*. The result can be summarized as follows:

THEOREM 3.1. *Suppose that Ω is a bounded region in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$ and $u(x,t)$ is a nonnegative classical solution of problem (1.1)-(1.3), and that the positive C^1-function h is such that*

$$h(\sigma) = b_1\sigma^p + b_2, \quad \sigma > 0, \tag{3.1}$$

with $b_1, b_2, p > 0$, and assume that the nonnegative integrable functions f and g satisfy the conditions

$$\xi f(\xi) \leq 2(1+\alpha)F(\xi), \quad \xi \geq 0, \tag{3.2}$$

$$\xi g(\xi) \geq 2(1+\gamma)G(\xi), \quad \xi \geq 0, \tag{3.3}$$

where

$$F(\xi) = \int_0^\xi f(s)ds, \quad G(\xi) = \int_0^\xi g(s)ds, \tag{3.4}$$

$$\gamma \geq max\{p, \alpha\}. \tag{3.5}$$

Let

$$\begin{aligned}\Theta(t) = 2\int_{\partial\Omega} G(u)dS - \int_\Omega \left(\frac{b_1}{1+p}|\nabla u|^{2p} + b_2\right)|\nabla u|^2 dx \\ - 2k(t)\int_\Omega F(u)dx,\end{aligned} \tag{3.6}$$

with $\Theta(0) > 0$, and let

$$k(t) > 0, \quad \frac{k'(t)}{k(t)} \leq -\beta, \quad t > 0, \tag{3.7}$$

for a positive constant β.

Then the solution $u(x,t)$ of problem (1.1)-(1.3) blows up at some finite time $t^* < T$ with

$$T = \frac{\Psi(0)}{2\gamma(1+\gamma)\Theta(0)}, \quad \gamma > 0,$$

where $\Psi(t) = \int_\Omega u^2 dx$ and $\Psi(0) > 0$. If $\gamma = 0$, then $T = \infty$.

REMARK 3.2. If we take $f(\xi) = a_1\xi^q$, $q \leq 2\alpha + 1$ and $g(\xi) = a_2\xi^r$, $r \geq 2\gamma + 1$, then the functions f and g satisfy the conditions (3.2) and (3.3).

Proof of Theorem 3.1. We compute the derivative

$$\begin{aligned}\Psi'(t) &= 2\int_\Omega u u_t dx = 2\int_\Omega u \mathrm{div}\left(h(|\nabla u|^2)\nabla u\right) dx - 2k(t)\int_\Omega u f(u) dx \\ &= 2\int_{\partial\Omega} u g(u) dS - 2\int_\Omega h(|\nabla u|^2)|\nabla u|^2 dx - 2k(t)\int_\Omega u f(u) dx.\end{aligned} \tag{3.8}$$

By using hypotheses (3.1)-(3.5), one can see that

$$\begin{aligned}\Psi'(t) &\geq 4(1+\gamma)\int_{\partial\Omega} G(u) dS - 2\int_\Omega \left(b_1|\nabla u|^{2p} + b_2\right)|\nabla u|^2 dx \\ &\quad - 4(1+\alpha)k(t)\int_\Omega F(u) dx \\ &\geq 2(1+\gamma)\Theta(t).\end{aligned} \tag{3.9}$$

Computing the derivative of $\Theta(t)$ in (3.6), it can be seen that

$$\begin{aligned}\Theta'(t) = {} & 2\int_{\partial\Omega} g(u) u_t dS - \int_\Omega \left(b_1|\nabla u|^{2p} + b_2\right)(|\nabla u|^2)_t dx \\ & - 2k(t)\int_\Omega f(u) u_t dx - 2k'(t)\int_\Omega F(u) dx.\end{aligned} \tag{3.10}$$

Integrating the equation

$$\begin{aligned}&\mathrm{div}\left(u_t(b_1|\nabla u|^{2p} + b_2)\nabla u\right) \\ &\qquad = u_t \mathrm{div}\left((b_1|\nabla u|^{2p} + b_2)\nabla u\right) + \frac{1}{2}(b_1|\nabla u|^{2p} + b_2)(|\nabla u|^2)_t,\end{aligned}$$

over Ω, we get

(3.11)

$$\begin{aligned}
&\int_\Omega \left(b_1|\nabla u|^{2p}+b_2\right)(|\nabla u|^2)_t dx\\
&\quad = 2\int_\Omega \operatorname{div}\left(u_t(b_1|\nabla u|^{2p}+b_2)\nabla u\right)dx - 2\int_\Omega u_t \operatorname{div}\left((b_1|\nabla u|^{2p}+b_2)\nabla u\right)dx\\
&\quad \le 2\int_{\partial\Omega} u_t g(u)dS - 2\int_\Omega u_t \operatorname{div}\left(h(|\nabla u|^2)\nabla u\right)dx.
\end{aligned}$$

Substituting (3.11) into (3.10), we have

(3.12)

$$\begin{aligned}
\Theta'(t) &\ge 2\int_\Omega u_t \operatorname{div}\left(h(|\nabla u|^2)\nabla u\right)dx - 2k(t)\int_\Omega u_t f(u)dx - 2k'(t)\int_\Omega F(u)dx\\
&= 2\int_\Omega u_t^2 dx - 2k'(t)\int_\Omega F(u)dx\\
&\ge 2\int_\Omega u_t^2 dx + 2\beta k(t)\int_\Omega F(u)dx \ge 2\int_\Omega u_t^2 dx \ge 0,
\end{aligned}$$

which implies $\Theta(t) > 0$ for all $t \in (0, t^*)$, since $\Theta(0) > 0$.

Making use of the Schwarz inequality, we can have the inequalities

$$2(1+\gamma)\Psi'(t)\Theta(t) \le (\Psi'(t))^2 = 4\left(\int_\Omega uu_t dx\right)^2 \le 2\Psi(t)\Theta'(t). \tag{3.13}$$

By (3.13), we can deduce

$$\begin{aligned}
(\Theta\Psi^{-(1+\gamma)})' &= \Theta'\Psi^{-(1+\gamma)} - (1+\gamma)\Theta\Psi^{-(\gamma+2)}\Psi'\\
&= \Psi^{-(\gamma+2)}[\Theta'\Psi - (1+\gamma)\Theta\Psi'] \ge 0.
\end{aligned}\tag{3.14}$$

Integrating (3.14) over $[0,t]$ and noticing that $\Psi(0) > 0$, one can see that

$$\Theta(t)\Psi^{-(1+\gamma)}(t) \ge \Theta(0)\Psi^{-(1+\gamma)}(0) =: M > 0;$$

that is,

$$\Theta(t) \ge M\Psi^{1+\gamma}(t). \tag{3.15}$$

It follows from (3.9) and (3.15) that

$$\Psi'(t) \ge 2(1+\gamma)\Theta(t) \ge 2M(1+\gamma)\Psi^{1+\gamma}(t). \tag{3.16}$$

If $\gamma > 0$, from (3.16) one can see that

$$(\Psi^{-\gamma})' = -\gamma\Psi^{-(\gamma+1)}\Psi' \le -2M\gamma(1+\gamma). \tag{3.17}$$

By (3.9), $\Theta(t) > 0$, and $\Psi(0) > 0$, we have the inequality

$$\Psi(t) > 0, \quad t > 0. \tag{3.18}$$

From (3.17) and (3.18), we get the inequality

$$0 < \Psi^{-\gamma}(t) \leq \Psi^{-\gamma}(0) - 2M\gamma(1+\gamma)t;$$

that is,

$$\Psi^{\gamma}(t) \geq \frac{1}{\Psi^{-\gamma}(0) - 2M\gamma(1+\gamma)t}.$$

If $t \to T = \frac{(\Psi(0))^{-\gamma}}{2M\gamma(1+\gamma)} = \frac{\Psi(0)}{2\gamma(1+\gamma)\Theta(0)}$, then $\Psi(t)$ will blow up at some time $t^* < T$. Consequently, one can see that

$$t^* \leq T = \frac{\Psi(0)}{2\gamma(1+\gamma)\Theta(0)}$$

is valid for all $\gamma > 0$.

If $\gamma = 0$, we have the inequalities

$$(\Theta\Psi^{-1})' \geq 0, \quad \Psi'(t) \geq 2M\Psi(t), \quad \Psi(t) \geq e^{2Mt}\Psi(0),$$

for all $t > 0$, which implies $t^* = \infty$. This completes the proof. $\square$

Remark 3.3. If the conditions in (3.7) are replaced by the following conditions:

$$k(t) < 0, \quad 0 < \frac{k'(t)}{k(t)} \leq \lambda, \quad t > 0, \quad \lambda > 0, \tag{3.19}$$

then one can easily obtain similar results as the ones in Theorem 3.1. In fact, (3.19) implies $k'(t) < 0$ and (3.12) becomes

$$\begin{aligned}\Theta'(t) &= 2\int_\Omega u_t^2 dx - 2k'(t)\int_\Omega F(u)dx \\ &\geq 2\int_\Omega u_t^2 dx \geq 0.\end{aligned}$$

4. Lower bounds for t^*

In this section, the domain $\Omega \subset R^N (N \geq 3)$ needs to be a convex bounded domain with smooth boundary. Moreover, we make some appropriate assumptions on nonlinear functions f and g to seek a lower bound for blow-up time t^*. We state our result below:

Theorem 4.1. *Suppose that $u(x,t)$ is the nonnegative classical solution of problem (1.1)-(1.3) and $u(x,t)$ blows up at t^*, and that the C^1-function $h(\sigma)$ is such that*

$$h(\sigma) \geq b_1\sigma^p + b_2, \quad \sigma > 0, \tag{4.1}$$

with $b_1, b_2, p > 0$, *and we assume that the nonnegative functions* f *and* g *are such that*

$$f(s) \geq a_1 s^q, \quad s \geq 0, \tag{4.2}$$

$$g(s) \leq a_2 s^r, \quad s \geq 0, \tag{4.3}$$

with $a_1, a_2, q, r > 1$ *and* $q + 1 \leq 2r$. *Define a function*

$$\varphi(t) := \int_\Omega u^m dx, \tag{4.4}$$

where m *is a parameter such that*

$$m > max\,\{4(N-2)(r-1), 2\}$$

and

$$k(t) \geq \eta, \quad t > 0, \quad \eta > 0. \tag{4.5}$$

Then the blow-up time t^* *is bounded below, i.e.,*

$$\int_{\varphi(0)}^{+\infty} \frac{d\zeta}{Q_1 \zeta^{\frac{3(N-2)}{3N-8}} + Q_2} < t^*,$$

where $\varphi(0) = \displaystyle\int_\Omega u_0^m dx$, *and* Q_1 *and* Q_2 *are positive constants to be determined later.*

Proof. From (4.1)-(4.4) and the divergence theorem, one can see that

$$\begin{aligned} \varphi^{'}(t) &= m \int_\Omega u^{m-1} u_t dx \\ &= m \int_\Omega u^{m-1} \text{div}\left(h(|\nabla u|^2)\nabla u\right) dx - mk(t) \int_\Omega u^{m-1} f(u) dx \\ &\leq -m(m-1)b_1 \int_\Omega u^{m-2} |\nabla u|^{2(p+1)} dx - m(m-1)b_2 \int_\Omega u^{m-2} |\nabla u|^2 dx \\ &\quad + ma_2 \int_{\partial\Omega} u^{m+r-1} dS - ma_1\eta \int_\Omega u^{m+q-1} dx. \end{aligned} \tag{4.6}$$

By Lemma 2.1, we can obtain the inequality

$$\int_{\partial\Omega} u^{m+r-1} dS \leq \frac{N}{\rho_0} \int_\Omega u^{m+r-1} dx + \frac{(m+r-1)d}{\rho_0} \int_\Omega u^{m+r-2} |\nabla u| dx. \tag{4.7}$$

It follows from Hölder's and Young's inequalities that

$$\int_\Omega u^{m+r-2} |\nabla u| dx$$

$$
\begin{aligned}
&= \int_\Omega u^{m+r-2-\frac{m-2}{2(p+1)}} u^{\frac{m-2}{2(p+1)}} |\nabla u| dx \\
&\leq \left[\int_\Omega u^{\frac{(m-2)(2p+1)+2r(p+1)}{2p+1}} dx\right]^{\frac{2p+1}{2(p+1)}} \left[\int_\Omega u^{m-2}|\nabla u|^{2(p+1)} dx\right]^{\frac{1}{2(p+1)}} \\
&\leq \frac{2p+1}{2(p+1)} \varepsilon_4^{-\frac{1}{2p+1}} \int_\Omega u^{\frac{(m-2)(2p+1)+2r(p+1)}{2p+1}} dx \\
&\qquad + \frac{1}{2(p+1)} \varepsilon_4 \int_\Omega u^{m-2} |\nabla u|^{2(p+1)} dx,
\end{aligned} \tag{4.8}
$$

where ε_4 is a positive constant to be determined later, and

$$
\begin{aligned}
\int_\Omega u^{\frac{(m-2)(2p+1)+2r(p+1)}{2p+1}} dx &\leq \left(\int_\Omega u^{m+2r-2} dx\right)^{P_1} |\Omega|^{1-P_1} \\
&\leq P_1 \int_\Omega u^{m+2r-2} dx + (1-P_1)|\Omega|,
\end{aligned} \tag{4.9}
$$

$$
P_1 = \frac{(m-2)(2p+1)+2r(p+1)}{(2p+1)(m+2r-2)} \in (0,1),
$$

$$
\begin{aligned}
\int_\Omega u^{m+r-1} dx &\leq \left(\int_\Omega u^{m+2r-2} dx\right)^{P_2} |\Omega|^{1-P_2} \\
&\leq P_2 \int_\Omega u^{m+2r-2} dx + (1-P_2)|\Omega|,
\end{aligned} \tag{4.10}
$$

$$
P_2 = \frac{m+r-1}{m+2r-2} \in (0,1).
$$

From (4.6)-(4.10), we get the inequality

$$
\begin{aligned}
\varphi'(t) \leq &\left[\frac{(m+r-1)dma_2}{2\rho_0(p+1)}\varepsilon_4 - m(m-1)b_1\right] \int_\Omega u^{m-2}|\nabla u|^{2(p+1)} dx \\
&+ ma_2 \left[\frac{NP_2}{\rho_0} + \frac{(2p+1)(m+r-1)dP_1}{2\rho_0(p+1)} \varepsilon_4^{-\frac{1}{2p+1}}\right] \int_\Omega u^{m+2r-2} dx \\
&+ ma_2 \left[\frac{N(1-P_2)}{\rho_0} + \frac{(2p+1)(m+r-1)d(1-P_1)}{2\rho_0(p+1)} \varepsilon_4^{-\frac{1}{2p+1}}\right] |\Omega| \\
&- m(m-1)b_2 \int_\Omega u^{m-2}|\nabla u|^2 dx - ma_1\eta \int_\Omega u^{m+q-1} dx.
\end{aligned} \tag{4.11}
$$

Choosing appropriate $\varepsilon_4 > 0$ such that

$$
\frac{(m+r-1)dma_2}{2\rho_0(p+1)} \varepsilon_4 = m(m-1)b_1,
$$

one can derive the inequality

$$
\begin{aligned}
\varphi'(t) \leq & -\frac{4(m-1)b_2}{m}\int_\Omega |\nabla u^{\frac{m}{2}}|dx + \widetilde{Q}_1\int_\Omega u^{m+2r-2}dx + \widetilde{Q}_2 \\
& - ma_1\eta\int_\Omega u^{m+q-1}dx.
\end{aligned} \tag{4.12}
$$

Here, we have used the identity

$$
|\nabla u^{\frac{m}{2}}| = \left(\frac{m}{2}\right)^2 u^{m-2}|\nabla u|^2,
$$

and

$$
\widetilde{Q}_1 = ma_2\left[\frac{NP_2}{\rho_0} + \frac{(2p+1)(m+r-1)dP_1}{2\rho_0(p+1)}\varepsilon_4^{-\frac{1}{2p+1}}\right] > 0,
$$

$$
\widetilde{Q}_2 = ma_2\left[\frac{N(1-P_2)}{\rho_0} + \frac{(2p+1)(m+r-1)d(1-P_1)}{2\rho_0(p+1)}\varepsilon_4^{-\frac{1}{2p+1}}\right]|\Omega| > 0.
$$

By Hölder's inequality, we have

$$
\begin{aligned}
\int_\Omega u^{m+q-1}dx &\geq |\Omega|^{-\frac{q-1}{m}}\left(\int_\Omega u^m dx\right)^{1+\frac{q-1}{m}} \\
&= |\Omega|^{-\frac{q-1}{m}}\varphi^{1+\frac{q-1}{m}}(t).
\end{aligned} \tag{4.13}
$$

Substituting (4.13) into (4.12), we get the inequality

$$
\begin{aligned}
\varphi'(t) \leq & -\frac{4(m-1)b_2}{m}\int_\Omega |\nabla u^{\frac{m}{2}}|^2dx + \widetilde{Q}_1\int_\Omega u^{m+2r-2}dx + \widetilde{Q}_2 \\
& - ma_1\eta|\Omega|^{-\frac{q-1}{m}}\varphi^{1+\frac{q-1}{m}}(t).
\end{aligned} \tag{4.14}
$$

We now consider the second term on the right-hand side of (4.14). By using Hölder's and Young's inequalities, we can obtain the inequalities

$$
\begin{aligned}
\int_\Omega u^{m+2r-2}dx &\leq \left[\int_\Omega u^{\frac{m(2N-3)}{2(N-2)}}dx\right]^{P_3}|\Omega|^{1-P_3} \\
&\leq P_3\int_\Omega u^{\frac{m(2N-3)}{2(N-2)}}dx + (1-P_3)|\Omega|,
\end{aligned} \tag{4.15}
$$

where

$$
P_3 = \frac{2(N-2)(m+2r-2)}{m(2N-3)} \in (0,1).
$$

It follows from (4.14) and (4.15) that

$$
\begin{aligned}
\varphi'(t) \leq & -\frac{4(m-1)b_2}{m}\int_\Omega |\nabla u^{\frac{m}{2}}|^2dx + \widetilde{Q}_1P_3\int_\Omega u^{\frac{m(2N-3)}{2(N-2)}}dx \\
& + \widetilde{Q}_1(1-P_3)|\Omega| + \widetilde{Q}_2 - ma_1\eta|\Omega|^{-\frac{q-1}{m}}\varphi^{1+\frac{q-1}{m}}(t).
\end{aligned} \tag{4.16}
$$

By applying Schwarz's inequality to the second term on the right side of (4.16), we have

$$\begin{aligned}\int_\Omega u^{\frac{m(2N-3)}{2(N-2)}}dx &\le \left(\int_\Omega u^m dx\right)^{\frac{1}{2}}\left(\int_\Omega u^{\frac{m(N-1)}{N-2}}dx\right)^{\frac{1}{2}}\\ &\le \left(\int_\Omega u^m dx\right)^{\frac{3}{4}}\left(\int_\Omega (u^{\frac{m}{2}})^{\frac{2N}{N-2}}dx\right)^{\frac{1}{4}}.\end{aligned}\tag{4.17}$$

To bound $\int_\Omega (u^{\frac{m}{2}})^{\frac{2N}{N-2}}dx$, we use the following Sobolev inequality given in [4]:

$$\begin{aligned}\|u^{\frac{m}{2}}\|^{\frac{N}{2(N-2)}}_{L^{\frac{2N}{N-2}}(\Omega)} &\le (c_s)^{\frac{N}{2(N-2)}}\|u^{\frac{m}{2}}\|^{\frac{N}{2(N-2)}}_{W^{1,2}(\Omega)}\\ &\le c\left(\|\nabla u^{\frac{m}{2}}\|^{\frac{N}{2(N-2)}}_{L^2(\Omega)}+\|u^{\frac{m}{2}}\|^{\frac{N}{2(N-2)}}_{L^2(\Omega)}\right),\quad N\ge 3,\end{aligned}\tag{4.18}$$

where c_s is a constant depending on Ω and N. By inserting (4.18) into (4.17), we can obtain the inequality

$$\begin{aligned}\int_\Omega u^{\frac{m(2N-3)}{2(N-2)}}dx \le &c\left(\int_\Omega u^m dx\right)^{\frac{3}{4}}\left(\int_\Omega |\nabla u^{\frac{m}{2}}|^2 dx\right)^{\frac{N}{4(N-2)}}\\ &+c\left(\int_\Omega u^m dx\right)^{\frac{2N-3}{2(N-2)}}.\end{aligned}\tag{4.19}$$

Now, we use Young's inequality to get the inequality

$$\begin{aligned}&\left(\int_\Omega u^m dx\right)^{\frac{3}{4}}\left(\int_\Omega |\nabla u^{\frac{m}{2}}|^2 dx\right)^{\frac{N}{4(N-2)}}\\ &\quad\le \frac{3N-8}{4(N-2)}\varepsilon_5^{-\frac{N}{3N-8}}\varphi^{\frac{3(N-2)}{3N-8}}(t)+\frac{N}{4(N-2)}\varepsilon_5\int_\Omega |\nabla u^{\frac{m}{2}}|^2 dx,\end{aligned}\tag{4.20}$$

where ε_5 is a positive constant to be determined later.

Applying Young's inequality to the second term on the right-hand side of (4.19), we get the inequality

$$\left(\int_\Omega u^m dx\right)^{\frac{2N-3}{2(N-2)}} \le P_4\varepsilon_6^{-\frac{P_5}{P_4}}\varphi^{\frac{3(N-2)}{3N-8}}(t)+P_5\varphi^{1+\frac{q-1}{m}}(t),\tag{4.21}$$

where

$$P_4=\frac{(3N-8)\left[m(2N-3)-2(N-2)(m+q-1)\right]}{2(N-2)\left[3m(N-2)-(m+q-1)(3N-8)\right]}\in(0,1),$$

$$P_5 = \frac{m\left[6(N-2)^2 - (2N-3)(3N-8)\right]}{2(N-2)\left[3m(N-2) - (m+q-1)(3N-8)\right]} \in (0,1),$$

and ε_6 is a positive constant to be determined later. From (4.16)-(4.21), we get the inequality

$$\varphi'(t) \le Q_1 \varphi^{\frac{3(N-2)}{3N-8}} + Q_2 + Q_3 \int_\Omega |\nabla u^{\frac{m}{2}}|^2 dx + Q_4 \varphi^{1+\frac{q-1}{m}},$$

where

$$\begin{aligned} Q_1 &= \widetilde{Q}_1 P_3 c \left[\frac{3N-8}{4(N-2)} \varepsilon_5^{-\frac{N}{3N-8}} + P_4 \varepsilon_6^{-\frac{P_5}{P_4}}\right], \\ Q_2 &= \widetilde{Q}_1 (1-P_3)|\Omega| + \widetilde{Q}_2, \\ Q_3 &= \frac{\widetilde{Q}_1 N P_3}{4(N-2)} c\varepsilon_5 - \frac{4(m-1)b_2}{m}, \text{ and} \\ Q_4 &= \widetilde{Q}_1 P_3 P_5 c \varepsilon_6 - m a_1 \eta |\Omega|^{-\frac{q-1}{m}}. \end{aligned}$$

Choosing appropriate $\varepsilon_5, \varepsilon_6 > 0$ so that Q_3 and Q_4 equal to zero, we can have the inequality

$$\varphi'(t) \le Q_1 \varphi^{\frac{3(N-2)}{3N-8}} + Q_2. \tag{4.22}$$

Integrating (4.22) from 0 to t, we get

$$\int_{\varphi(0)}^{\varphi(t)} \frac{d\zeta}{Q_1 \zeta^{\frac{3(N-2)}{3N-8}} + Q_2} \le t.$$

Letting $t \to t^*$, we get the result

$$\int_{\varphi(0)}^{+\infty} \frac{d\zeta}{Q_1 \zeta^{\frac{3(N-2)}{3N-8}} + Q_2} \le t^*.$$

This completes the proof. □

Remark 4.2. In fact, if we consider the more general quasilinear diffusion equation with time-dependent coefficients

$$\frac{1}{k_1(t)} u_t = k_2(t) \operatorname{div}\left(h(|\nabla u|^2)\nabla u\right) - k_3(t) f(u), \quad (x,t) \in \Omega \times (0, t^*), \tag{4.23}$$

where $k_1(t)$, $k_2(t)$, and $k_3(t)$ are positive differentiable functions, then, as in [19], we can replace the time variable by the new variable

$$z(t) = \int_0^t k_1(\tau) k_2(\tau) d\tau.$$

Then the differential equation in (4.23) can be reduced to the following form:

$$u_z = \operatorname{div}\left(h(|\nabla u|^2)\nabla u\right) - K(z)f(u), \quad (x,z) \in \Omega \times (0, z^*),$$

where $z^* = z(t^*)$ and $K(z) = \dfrac{k_3(t(z))}{k_2(t(z))}$. Under appropriate assumptions, the results given in Sections 2, 3, and 4 are therefore valid to the more general problem (4.23).

5. Applications

In this section, we present two examples to demonstrate applications of Theorems 3.1 and 4.1.

EXAMPLE 5.1. *Let $u(x,t)$ be a solution of the following problem:*

$$u_t = \operatorname{div}\left((1+|\nabla u|)\nabla u\right) - \frac{3}{2}e^{-2t}u^{\frac{1}{6}}, \quad (x,t) \in \Omega \times (0,t^*),$$

$$(1+|\nabla u|)\frac{\partial u}{\partial \nu} = u^2, \quad (x,t) \in \partial\Omega \times (0,t^*),$$

$$u(x,0) = u_0(x) = |x| + \sqrt{2} - 1 > 0, \quad x \in \Omega,$$

where $\Omega = \left\{x = (x_1,x_2,x_3) \,|\, |x|^2 = \sum_{i=1}^3 x_i^2 < 1\right\}$. Then we have

$$h(|\nabla u|^2) = 1 + |\nabla u|, \quad k(t) = e^{-2t}, \quad f(u) = \frac{3}{2}u^{\frac{1}{6}}, \quad g(u) = u^2.$$

Now, we set $\beta = 2$, $a_1 = \frac{3}{2}$, $a_2 = 1$, and $\gamma = \frac{1}{2}$. Then it is easy to verify that (3.2)-(3.5) hold. By (3.6), one can see that

$$\begin{aligned}\Theta(0) = {}& 2\int_{\partial\Omega}\left(\int_0^{u_0} s^2 ds\right) dS - \int_\Omega \left(\frac{1}{1+\frac{1}{2}}|\nabla u_0| + 1\right)|\nabla u_0|^2 dx \\ & - 3k(0)\int_\Omega\left(\int_0^{u_0} s^{\frac{1}{6}} ds\right) dx = 3.83 > 0.\end{aligned}$$

It follows from Theorem 3.1 that $u(x,t)$ blows up in finite time t^*, and we have

$$t^* < T = \frac{\Psi(0)}{2\gamma(1+\gamma)\Theta(0)} = 1.01,$$

where $\Psi(0) = \int_\Omega u_0^2 dx = 5.83.$

EXAMPLE 5.2. *Let $u(x,t)$ be a solution of the following problem:*

$$u_t = \operatorname{div}\left((6+100|\nabla u|)\,\nabla u\right) - 6(t+1)u^2, \quad (x,t) \in \Omega \times (0,t^*),$$

$$(6+100|\nabla u|)\,\frac{\partial u}{\partial \nu} = \frac{2}{5}u^{\frac{3}{2}}, \quad (x,t) \in \partial\Omega \times (0,t^*),$$

$$u(x,0) = u_0(x) = |x|^4 + 15.99 \times 10^{-2} > 0, \quad x \in \Omega,$$

where $\Omega = \left\{ x = (x_1, x_2, x_3) \,|\, |x|^2 = \sum_{i=1}^3 x_i^2 < (\frac{1}{10})^2 \right\}$. *Then we have*

$$h(|\nabla u|^2) = 6 + 100|\nabla u|, \quad k(t) = 4(1+t), \quad f(u) = \frac{3}{2}u^2, \quad g(u) = \frac{2}{5}u^{\frac{3}{2}},$$

and choosing $a_1 = \frac{3}{2}, a_2 = \frac{2}{5}, \eta = 4$, *and* $m = 3$, *it can be easily seen that (3.2)-(3.5) hold and*

$$\varepsilon_4 = \frac{3000}{7}, \quad \varepsilon_5 = 0.45, \quad \varepsilon_6 = 2.64,$$

$$Q_1 = 125.44, \quad Q_2 = 0.03,$$

$$\varphi(0) = \int_\Omega u_0^2 dx = 1.71 \times 10^{-4}.$$

It then follows from Theorem 4.1 that

$$t^* \geq \int_{1.71\times 10^{-4}}^{+\infty} \frac{d\zeta}{125.44\zeta^3 + 0.03} = 2.50.$$

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References

[1] K. Baghaei and M. Hesaaraki, *Lower bounds for the blow-up time in the higher-dimensional nonlinear divergence form parabolic equations*, C. R. Acad. Sci. Paris. Ser. I. **351** (2013), 731-735.

[2] C. Bandle and H. Brunner, *Blow-up in diffusion equations: A survey*, J. Comput. Appl. Math. **97** (1998), 3-22.

[3] J. Bebernes and D. Eberly, *Mathematical Problems from Combustion Theory*, Springer-Verlag, New York, 1989.

[4] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer-Verlag, New York, 2011.

[5] C. Enache, *Blow-up phenomena for a class of quasilinear parabolic problems under Robin boundary condition*, Appl. Math. Lett. **24** (2011), 288-292.

[6] Z. B. Fang and Y. Chai, *Blow-up analysis for a quasilinear parabolic equation with inner absorption and nonlinear Neumann boundary condition*, Abstr. Appl. Anal. **2014**, Article ID289245, (2014), 8 pages.

[7] Z. B. Fang, R. Yang, and Y. Chai, *Lower bounds estimate for the blow-up time of a slow diffusion equation with nonlocal source and inner absorption*, Math. Probl. Eng. **2014** Article ID764248, (2014), 6 pages.

[8] Z. B. Fang and Y. X. Wang, *Blow-up analysis for a semilinear parabolic equation with time-dependent coefficients under nonlinear boundary flux*, Z. Angew. Math. Phys. **66** (2015), 2525-2541.

[9] J. Filo, *Diffusivity versus absorption through the boundary*, J. Diff. Eq. **99** (1992), 281-305.

[10] V. A. Galaktionov and J. L. Vázquez, *The problem of blow up in nonlinear parabolic equations*, Discrete Contin. Dyn. Syst. **8** (2002), 399-433.

[11] H. A. Levine, *Nonexistence of global weak solutions to some properly and improperly posed problems of mathematical physics: The method of unbounded Fourier coefficients*, Math. Ann. **214** (1975), 205-220.

[12] H. A. Levine, *The role of critical exponents in blow-up theorems*, SIAM Rev. **32** (1990), 262-288.

[13] H. A. Levine and L. E. Payne, *Nonexistence theorems for the heat equation with nonlinear boundary conditions and for porous medium equation backward in time*, J. Diff. Eq. **16** (1974), 319-334.

[14] Y. F. Li, Y. Liu, and C. H. Lin, *Blow-up phenomena for some nonlinear parabolic problems under mixed boundary conditions*, Nonlinear Anal. RWA, **11** (2010), 3815-3823.

[15] M. Marras and S. Vernier Piro, *On global existence and bounds for blow-up time in nonlinear parabolic problems with time dependent coefficients*, Discrete Cont. Dyn. S. Supplement, **2013**(special), (2013), 535-544.

[16] C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, Plenum Press, New York, 1992.

[17] L. E. Payne and G. A. Philippin, *Blow-up in a class of non-linear parabolic problems with time dependent coefficients under Robin type boundary conditions*, Appl. Anal. **91** (2012), 2245-2256.

[18] L. E. Payne and G. A. Philippin, *Blow-up phenonmena in parabolic problems with time dependent coefficients under Neumann Boundary conditions*, P. Roy. Soc. Edinb. A. (2012), 625-631.

[19] L. E. Payne and G. A. Philippin, *Blow-up phenonmena in parabolic problems with time dependent coefficients under Dirichlet Boundary conditions*, P. Am. Math. Soc. **141** (2013), 2309-2318.

[20] L. E. Payne, G. A. Philippin, and S. V. Piro, *Blow-up phenonmena for a semilinear heat equation with nonlinear boundary condition. II*, Nonlinear Anal. **73** (2010), 971-978.

[21] L. E. Payne, G. A. Philippin, and P. W. Schaefer, *Blow-up phenomena for some nonlinear parabolic problems*, Nonlinear Anal. **69** (2008), 3495-3502.

[22] R. Quittner and P. Souplet, *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*, Birkhauser Advanced Texts, Basel, 2007.

[23] A. A. Samarskii, V. A. Galaktionov, S. P.Kurdyumov, and A. P. Mikhailov, *Blow-Up in Quasilinear Parabolic Equations. Walter de Gruyter*, Berlin, 1995.

[24] B. Straughan, *Explosive Instabilities in Mechanics*, Springer, Berlin, 1998.
[25] J. L. Vázquez, *The Porous Medium Equations: Mathematical Theory*, Oxford: Oxford University Press, Oxford, 2007.

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